MODULAR SYMBOLS AND L-FUNCTIONS

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1. Introduction

As the first non-overview lecture in this seminar, we will be setting up a lot of notation, getting comfortable working with modular symbols, and then hopefully discussing some of the major inputs which make the theory work. The first half of the talk we will work through the example of the unique cusp form of weight two and level 11 on $X_0(11)$. In the second half, we will bring in the theoretical results, which explain why modular symbols work. The two main results to pay attention to are Th'm 6.2 and Prop 7.6. We include a long discussion about cohomology for example, which won't be necessary in subsequent talks.

The relevant material we intend to cover is in Section 1.1 of [BDInv] and Section 1.1 of [BDAnn]. We attempt to follow as closely as possible the notation used in these two papers. The proofs of the formulae can be found in [MTT] though the notation is different there. We made extensive use of Rob Pollack and Glenn Stevens' Notes from the AWS [PoSt] as well as unpublished notes of Bellaiche from a course taught at Brandeis [Bel]. We also used an unpublished manuscript of Brian Conrad's on Modular Forms and Galois representations [Con].

2. An Example

Let $\mathcal{H}$ denote the complex upper half plane. Let $q = e^{2\pi i z}$ be the standard exponential function on $\mathcal{H}$. Consider the function

$$f(z) = q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{11n})^2,$$

which is the unique cusp form of weight 2 and level $\Gamma_0(11)$. Recall that $\Gamma_0(N)$ is the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ consisting of matrices with lower-left entry divisible by N.

Let $Y_0(11)$ denote the open modular curve $\mathcal{H}/\Gamma_0(11)$ over $\mathbb{C}$. The compactification of this modular curve will be denoted by $X_0(11)$. Topologically, we can identify $X_0(11)$ with the quotient of $\mathcal{H}^* = \mathcal{H}\cup\mathbb{P}^1(\mathbb{Q})$ by the action of $\Gamma_0(11)$ (one can put a topology on $\mathcal{H}^*$ such that $\mathrm{SL}_2(\mathbb{Z})$ acts through homeomorphisms).

Topologically, $X_0(11)$ is the quotient of the $\mathcal{H}\cup P^1(\mathbb{Q})$ modulo the action of $\Gamma_0(11)$. One can also make sense of a complex structure on this quotient, but I won't go into how one does that. If you want more details on the very real subtleties involved in working with modular curves, I refer you to the handouts from Brian's course on modular curves which as of this writing is still available at http://math.stanford.edu/ conrad/248BPage/. I will for the most part gloss over these issues, because that is not the focus of this seminar.

Now, the differential form $\omega_f = f dz$ on $\mathcal{H}$ is invariant under the action of $\Gamma_0(11)$ and so gives rise to a differential form on $Y_0(11)$. Because f is a cusp form, ω_f in fact extends to a global differential form on $X_0(11)$, that is, an element of $H^0(X_0(11), \Omega^1_{X_0(11)})$.

We have a 1-form on a Riemann surface, so one interesting thing to do is to integrate against integral homology classes. There is a convenient choice of base point, namely the image of $i\infty$ on $X_0(11)$, which we will just denote by ∞. Let $\gamma \in \Gamma_0(11)$ and take any continuous path from ∞ to $\gamma(\infty)$ in $\overline{H}$. This path clearly projects to a loop in $X_0(11)$ based at ∞. The homology class is independent of path and will be denoted by $\{\infty \to \gamma(\infty)\}$.

We define

$$\widetilde{I}_f\{\infty \to \gamma(\infty)\} := 2\pi i \int_{\infty}^{\gamma(\infty)} f dz.$$

This is called a period of f. Following Samit's notation, let $\Lambda_f = \{\widetilde{I}_f\{\infty \to \gamma(\infty)\} | \gamma \in \Gamma_0(11)\} \subset \mathbb{C}$. It is non-trivial fact that Λ_f is a lattice in $\mathbb{C}$ whenever f is a Hecke eigenform defined over $\mathbb{Q}$. In this situation, it is not so difficult to see. The modular curve $X_0(11)$ has genus 1 and so $H_1(X_0(11), \mathbb{Z})$ has rank 2 over $\mathbb{Z}$.

With a little help from SAGE, we find that two generators for the homology are given by $\{\infty \to \gamma_i(\infty)\}$ with

$$\gamma_1 = \begin{pmatrix} 8 & 1 \\ -33 & -4 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 9 & 1 \\ -55 & -6 \end{pmatrix}.$$

A simpler way is to just say the paths given by $\{\frac{-1}{8} \to 0\}$ and $\{\frac{-1}{9} \to 0\}$, but we wanted to be based at ∞ so we stated it this other way.

Remark 2.1. For any $\gamma \in \Gamma_0(11)$ and any $\tau \in \mathcal{H}^*$, the value of

$$2\pi i \int_{\tau}^{\gamma(\tau)} f dz$$

is in fact independent of τ. This follows from an easy computation using the $\Gamma_0(11)$ invariance of ω_f. This is an important point for actual computation, because one has to make a good choice of basepoint for things to converge rapidly.

Making a clever choice of basepoint, we compute the periods of f along the path given by γ_1 and γ_2 which turn out to be:

$$\{1.2692093043, .63460465214 - 1.4588166169i\}$$

These are clearly $\mathbb{R}$-linearly independent so they span a lattice in $\mathbb{C}$ and hence define an elliptic curve over $\mathbb{C}$! Which one????? Using the basis above, we can compute the j-invariant to be

$$-757.6726378$$

up to a certain number of digits. Unfortunately, this does not look much like an integer and so it seems a bit tricky where to go from here. To say much more, we need to know that a priori, this elliptic curve should be defined over $\mathbb{Q}$ and furthermore have some control over the denominators can appear in the rational j-invariant. The more refined theory of modular curves over $\mathbb{Q}$ realizes this map as a map of curves over $\mathbb{Q}$, and we in fact know that the elliptic curve should have good reduction away from 11. With this information (and a little bit more), we find that

$$-757.6726378 \approx -\frac{2^{12}31^3}{11^5}$$

which is the actual j-invariant. In the Cremona labels, this is the curve '11a1'. The minimal Weierstrauss equation for this curve, which we call E_f, is

$$y^2 + y = x^3 - x^2 - 10x - 20.$$

While we are here, we might as well compute some more invariants attached to E_f and see how they are reflected in f. As we will see in the next section, it is convenient not to restrict to integrating only along paths of the form $\infty \to \gamma(\infty)$. One particularly nice motivation is that special values of L-functions are often encoded in these types of integrals. We will discuss this in far more detail near the end of the piece.

For now, we focus on one particular integral

$$2\pi i \int_\infty^0 f(z)dz = \widetilde{I}_f(\{\infty \to 0\}).$$

The classical formula writes the L-function of a modular form in terms of its Mellin transform which we will return to later.

In any case,

$$L(f,1) = L(E_f,1) \approx 0.253841860855911.$$

Very enlightening right, but wait, if we divide this value by the value of the real period which we computed earlier

$$L(f,1)/1.2692093043 \approx .2000000000002$$

which looks darn close to 1/5. There are two different interesting phenomenon going on here. First, the new period $\widetilde{I}_f(\{\infty \to 0\})$ is not in the lattice generated by the other periods, however, 5 times it is. This is a general fact, which hopefully we will have a opportunity to discuss later. The second question one could ask is what does the 5 mean. The BSD conjectures say that

$$L(E_f, 1) = \frac{\Omega_E^+ \mathrm{Sha}(E) R_E \prod c_p}{|E_{\mathrm{tors}}|^2}.$$

The only non-integral term in the formula is Ω_E^+ which in our situation is $1.269209\ldots$. The torsion subgroup of E_f does in fact have order 5. One power of 5 gets cancelled by the Tamagawa number at 11 which is also 5. The moral here being that the periods of a modular form encode a lot of interesting information.

For more general modular curves, say $X_0(N)$, things do get more complicated as the size of the genus grows. However, there has been a lot of work done on computing periods of modular forms. I recommend William Stein's book "Modular Forms, a Computational Approach" for a gentle and readable introduction. In a nutshell, modular symbols provide a purely combinatorial way of computing a basis for the homology of the modular curves. Here are a couple more examples for fun:

For $X_0(17)$ of genus 1, we have the two paths

$$\{\frac{-1}{14} \to 0\}, \{\frac{-1}{15} \to 0\}$$

For $X_0(15)$ of genus 1, we have two paths

$$\{\frac{1}{4} \to \frac{1}{3}\} - \{-\frac{1}{2} \to -\frac{2}{5}, \{0 \to \frac{1}{5}\} - \{-\frac{1}{2} \to -\frac{2}{5}\}.$$

For $X_0(23)$ of genus 2, we have

$$\{\frac{-1}{17} \to 0\}, \{\frac{-1}{19} \to 0\}, \{\frac{-1}{20} \to 0\}, \{\frac{-1}{21} \to 0\}.$$

3. Modular Symbols

As we have mentioned before, modular symbols provide a combinatorial way of computing with modular forms. They are not much more than homology in disguise. They are nevertheless a useful framework to work in. The periods is what we are really interested in as you will see. In the remainder of the piece, we will used F to denote a field of characteristic 0, which one can usually think of as $\mathbb{Q}, \mathbb{R}, \mathbb{C}$. We will often make statements which are true over any of those fields so we would rather not have to write out all three every time.

Let Δ be the free abelian group on $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \infty$. We write element of Δ as a finite sum $\sum n_i\{r_i\}$ where $r_i \in \mathbb{Q} \cup \infty$. This has an $\mathbb{Z}$-linear action of $\mathrm{GL}_2(\mathbb{Q})$ given by

$$\gamma(\{r\}) = \{\frac{ar+b}{cr+d}\}$$

if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

We have a "trace" map

$$\Delta \to \mathbb{Z}$$

given by summing over the coefficients, whose kernel Δ_0 is the degree 0 elements of Δ.

Remark 3.1. The submodule Δ_0 is generated by elements of the form $\{s\} - \{r\}$. One should think of these elements as "representing" a geodesic path from r to s in $\mathcal{H}^*$. In recognition of this insight, we will also write $\{s\} - \{r\}$ as $\{r \to s\}$. Note that $\{r \to s\} + \{s \to t\} = \{r \to t\}$ in Δ_0. This is the relation of composing paths.

Example 3.2. Let f be a weight 2 cusp form on level $\Gamma_0(N)$. We can define a homomorphism from $\widetilde{I}_f : \Delta_0 \to \mathbb{C}$ defined by

$$\widetilde{I}_f(\{r \to s\}) := 2\pi i \int_r^s f dz,$$

and extended linearly. The integral is independent of path since f is holomorphic and $\mathcal{H}^*$ is simply-connected.

Remark 3.3. When we evaluate $\widetilde{I}_f$ on elements of Δ_0, we may get slightly more than just the periods of f. Not all elements of Δ_0 are equivalent to one of the form $\{r \to \gamma(r)\}$ for $\gamma \in \Gamma_0(N)$. What you get is actually not too much more, it is a result of ??? that $\widetilde{I}_f(\Delta_0)$ is contained in the $\mathbb{Q}$ span of the periods of f. The denominators that show up are related to the torsion subgroup of E_f. We saw that in the introductory example.

Before we further elaborate the properties of $\widetilde{I}_f$, we need to introduce periods for higher weight forms. Let $S_k(\Gamma_0(N), \mathbb{C})$ denote the space of cusp forms of even weight $k \geq 2$ and level $\Gamma_0(N)$. Let $f \in S_k(\Gamma_0(N), \mathbb{C})$.

If $k > 2$, $f dz$ no longer has the desired invariance property. To remedy this, we introduce a new factor.

Definition 3.4. Let F be a field of characteristic 0. Define $\mathcal{P}_k(F)$ to be the F-vector space of polynomials in X, Y of degree $k-2$. Define an right-action of $\mathrm{GL}_2(\mathbb{Q})$ on $\mathcal{P}_k(F)$ given by

$$(P|_\gamma)(X,Y) = \det(\gamma)^{-(k-2)/2} P(aX + bY, cX + dY)$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Warning: In [BDInv], they do not include the determinant factor when defining the slash operation on $\mathcal{P}_k(F)$. This is not a problem for $\mathrm{SL}_2(\mathbb{Q})$ action, but one has to watch out for factors of p showing up (or not showing up) in formulas related to Hecke operators.

Definition 3.5. Let F be a field of characteristic 0. Define $V_k(F)$ to be the F-linear dual $\mathrm{Hom}_F(\mathcal{P}_k(F), F)$ to the space of $(k-2)$-homogeneous polynomials. We endow $V_k(F)$ with a *left* $\mathrm{GL}_2(\mathbb{Q})$ action given by

$$\gamma.\Psi(P) = \Psi(P|_\gamma).$$

In the context of modular forms, one often sees $\mathcal{P}_k(F)$ written as $\mathrm{Sym}^{k-2}(F^2)^*$ and $V_k(F)$ written as $\mathrm{Sym}^{k-2}(F^2)$. This perspective will arise when we relate modular symbols to cohomology in the next section so one should keep it in mind.

The analogue of the period morphism for higher weight forms does not take values in $\mathbb{C}$, but instead takes values in $V_k(\mathbb{C})$. If $f \in S_k(\Gamma_0(N), \mathbb{C})$, we define

$$\widetilde{I}_f\{r \to s\} \in V_k(\mathbb{C})$$

by

$$\widetilde{I}_f\{r \to s\}(P) := 2\pi i \int_r^s g(z) P(z,1) dz.$$

Different authors take different perspectives on the above map, and we will undoubtably encounter them all. Our perspective is to think of $\widetilde{I}_f \in \mathrm{Hom}(\Delta_0, V_k(\mathbb{C}))$. Another perspective is to think of $\widetilde{I}_f$ as a $\mathbb{C}$-bilinear form on

$$\Delta_0 \times \mathcal{P}_k(\mathbb{C}).$$

If we want to think about changing f, then we might say that $\widetilde{I}$ is a tri-linear form on $S_k(\Gamma_0(N)) \times \Delta_0 \times \mathcal{P}_k(\mathbb{C})$, or even a $V_k(\mathbb{C})$-valued bilinear form on $S_k(\Gamma_0(N)) \times \Delta_0$. As we discover more properties of these "modular symbols" different perspectives will be more convenient than other for stating them, but they are all different forms of the same thing.

Before we prove our first property of $\widetilde{I}_f$, we note a small lemma about the action on $\mathcal{P}_k(F)$.

Lemma 3.6. *Let $P(x, y) \in \mathcal{P}_k(F)$ and $\gamma \in \mathrm{GL}_2(\mathbb{Q})$. Then,*

$$P(\gamma(z), 1) = \det(\gamma)^{(k-2)/2} (cz + d)^{-(k-2)} P|_\gamma(z, 1).$$

Proof. The polynomial P is homogeneous of degree $k-2$. Using the homogeneity, one sees that

$$(cz+d)^{k-2}P(\frac{az+b}{cz+d},1) = P(az+b, cz+d) = \det(\gamma)^{(k-2)/2}P|_{\gamma}(z,1).$$

Then, just divide through by $(cz+d)^{k-2}$. □

The next Proposition says that $\widetilde{I}_f$ is in fact $\Gamma_0(N)$-equivariant.

Proposition 3.7. *Let $f \in S_k(\Gamma_0(N), \mathbb{C})$. Then, $\widetilde{I}_g$ is an element of $Hom_{\Gamma}(\Delta_0, V_k(\mathbb{C}))$.*

Proof. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $\Gamma_0(N)$, and let $P \in \mathcal{P}_k(\mathbb{C})$. Then,

$$\begin{aligned}
\widetilde{I}_f\{\gamma(r) \to \gamma(s)\}(P) &= \int_{\gamma(r)}^{\gamma(s)} g(z)P(z,1)dz \\
&= \int_r^s g(\gamma(z))P(\gamma(z),1)d(\gamma(z)) \\
&= \int_r^s [(cz+d)^k g(z)] * [(cz+d)^{-(k-2)}P|_{\gamma}(z,1)] * [(cz+d)^{-2}dz] \\
&= \int_r^s g(z)P|_{\gamma}(z,1)dz.
\end{aligned}$$

where we use Lemma 3.6 in the second to last step. □

Definition 3.8. If Γ is a any congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, we define the space of *modular symbols of level* Γ with values in M to be $\mathrm{Hom}_{\Gamma}(\Delta_0, M)$, where M is any left Γ-module. Call this space $\mathrm{MS}_{\Gamma}(M)$.

The previous Proposition showed that to any modular form f, one can attach a modular symbol $\widetilde{I}_f$ valued in $V_k(\mathbb{C})$ of specified level. It will be very important later although $\widetilde{I}_f$ only makes sense over $\mathbb{C}$ the space of modular symbols with values in V_k makes perfect sense over $\mathbb{Q}$, for example. This will give rise another "canonical" $\mathbb{Q}$ structure on the space of modular forms.

We record one more result before turning to cohomology.

Proposition 3.9. *(Base Change) Assume M is an $R[\Gamma]$-module. Let $R \to R'$ be any flat ring extension. Then, there is a natural isomorphism*

$$\mathrm{MS}_{\Gamma}(M) \otimes_R R' \cong \mathrm{MS}_{\Gamma}(M_{R'}).$$

In particular, $\mathrm{MS}_{\Gamma}(V_k(\mathbb{Q})) \otimes \mathbb{C} = \mathrm{MS}_{\Gamma}(V_k(\mathbb{C}))$.

Proof. We have that

$$\mathrm{Hom}_{\mathbb{Z}}(\Delta_0, M) \otimes_R R' \cong \mathrm{Hom}_{\mathbb{Z}}(\Delta_0, M_{R'})$$

so the issue is with taking Γ-invariants. The difficulty here is that Γ is infinite, otherwise, we could easily write $\mathrm{Hom}_\Gamma(\Delta_0, M)$ as the kernel of some R-morphism and deduce the result from there. To get around that, we use a result of Manin (§1.7 [Man72] or Th'm 3.13 [St]) which says that Δ_0 is finitely presented as a $\mathbb{Z}[\Gamma]$-module, that is, there exists an exact sequence

$$\mathbb{Z}[\Gamma]^r \to \mathbb{Z}[\Gamma]^s \to \Delta_0 \to 0$$

of Γ-modules. Using this, we get

$$0 \to \mathrm{Hom}_\Gamma(\Delta_0, M) \to \mathrm{Hom}_\Gamma(\mathbb{Z}[\Gamma]^s, M) \to \mathrm{Hom}_\Gamma(\mathbb{Z}[\Gamma]^r, M)$$

and the "universal" property of $\mathbb{Z}[\Gamma]$ says that $\mathrm{Hom}_\Gamma(\mathbb{Z}[\Gamma], M) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, M)$. The formation of this sequence commutes with base change and flatness ensures the kernel commutes with base change. □

4. Relationship to Cohomology

We would like to relate the space of modular symbols $\mathrm{MS}_\Gamma(V_k(\mathbb{C}))$ to cohomology of the modular curve. Before we do, we review the classical Eichler-Shimura isomorphism.

4.1. **Eichler-Shimura.** In this section, we give an abbreviated summary of the contents of Section §3.4 in Brian's currently unavailable book on "Modular forms, cohomology, and the Ramanujan conjecture." Assume for this section that Γ is a congruence subgroup acting freely on $\mathcal{H}$. This is extremely useful for cohomological arguments. Later, we will show that we can deduce everything we need from this case.

Let $S_k(\Gamma, \mathbb{C})$ be the space of modular forms of weight k and level Γ. Let $Y_\Gamma := \mathcal{H}/\Gamma$ be the modular curve of level Γ. Note that since the action is free $\mathcal{H} \to Y_\Gamma$ is the universal cover of Y_Γ.

Let $V_k(\mathbb{R})$ as before be dual to homogeneous polynomials of degree $k-2$. We want to construct a local system of $\mathbb{R}$-vector spaces on Y_Γ. Thinking of $V_k(\mathbb{R})$ as the constant sheaf on $\mathcal{H}$, we can descend to a sheaf on Y_Γ, if you want $\mathcal{H} \times V_k(\mathbb{R})/\Gamma$. Denote this by $\underline{V}_k(\mathbb{R})$.

Remark 4.1. For other purposes, it is useful to connect up the local system $\underline{V}_k(\mathbb{R})$ with respresentations of the fundamental group $\pi(Y_\Gamma, z_0)$. One has to be careful to note that $\pi(Y_\Gamma, z_0) \cong \Gamma^{\mathrm{opp}}$ and pay careful attention to left and right actions. Brian does this quite carefully in his notes.

One can associate to any $f \in S_k(\Gamma, \mathbb{C})$ an element $\mathrm{Sh}_\Gamma(f) \in H^1(Y_\Gamma, \underline{V}_k(\mathbb{C}))$. In weight two, the map is given by taking the image of ω_f in the map from de Rham cohomology to Hodge cohomology.

In more concrete terms, the map in weight two is integrating ω_f over cycles. In higher weight, one similar uses an edge map in the Hodge to De Rham spectral sequence. This can be described fairly explictly, but we hold off to give a group cohomological interpretation later which will be more useful for our purposes.

We need one more preliminary before stating the theorem. For a "nice" enough manifold Y(like Y_Γ), there is a natural map

$$H^q_c(Y, M) \to H^q(Y, M)$$

for any local system M, which extends the map in degree 0 corresponding to the inclusion of compactly supported global sections into global sections.

Definition 4.2. The image of compact cohomology in ordinary cohomology is denoted by $H^q_!(Y, M)$. It goes by many names one of which is *parabolic* cohomology.

Though $H^q_!$ is not a derived functor, it has many nice properties which can be deduced from the properties of H^q_c and H^q.

Theorem 4.3. *(Eichler-Shimura). The map* $\mathrm{Sh}_\Gamma : S_k(\Gamma, \mathbb{C}) \to H^1(Y_\Gamma, \underline{V}_k(\mathbb{C}))$ *lands in parabolic cohomology* $H^1_!(Y_\Gamma, \underline{V}_k(\mathbb{C}))$*. The resultiing map*

$$\mathrm{Sh}_\Gamma \oplus \overline{\mathrm{Sh}}_\Gamma : S_k(\Gamma, \mathbb{C}) \oplus \overline{S_k(\Gamma, \mathbb{C})} \to H^1_!(Y_\Gamma, \underline{V}_k(\mathbb{C})).$$

is an isomorphism, which is Hecke-compatible.

There are essentially three parts to the proof:

(1) Show that the image of $S_k(\Gamma, \mathbb{C})$ lands in $H^1_!$.
(2) Show that the map

$$\mathrm{Re}(\mathrm{Sh}_\Gamma) : S_k(\Gamma, \mathbb{C}) \to H_!(Y_\Gamma, \underline{V}_k(\mathbb{R}))$$

is injective.
(3) Compute the real dimension of both sides and show that they match up.

We will focus on (1) and give an indication of how (2) is proved.

Proposition 4.4. *The image of* Sh_Γ *lands in* $H^1_!(Y_\Gamma, \underline{V}_k(\mathbb{C}))$*.*

To prove the Proposition, we first need to a way to get our hands on $H^1_!$. For now, we let $\mathcal{F}$ be any local system of $\mathbb{R}$-vectors spaces on a nice topological space Y.

It is standard fact in algebraic topology, which is also true with coefficients in a local system that

$$H^i_c(Y, \mathcal{F}) = \lim_{\to K} H^i_K(Y, \mathcal{F}),$$

where H^i_K is the derived functor of the global sections with support in K a compact set. The compact sets are ordered by inclusion. We also recall that for any particular compact K, we have an exact sequence

$$\cdot \to H^0(Y,\mathcal{F}) \to H^0(Y-K,\mathcal{F}) \to H^1_K(Y,\mathcal{F}) \to H^1(Y,\mathcal{F}) \to H^1(Y-K,\mathcal{F}) \to \cdot.$$

We can choose a cofinal directed system of compact K and take the direct limit of the above sequence to get the following exact sequence

$$\cdot \to H^0(Y,\mathcal{F}) \to \lim_{\to K} H^0(Y-K,\mathcal{F}) \to H^1_c(Y,\mathcal{F}) \to H^1(Y,\mathcal{F}) \to \lim_{\to K} H^1(Y-K,\mathcal{F}) \to \cdot.$$

In particular, we can identify $H^1_!$ as the kernel in the left exact sequence

$$0 \to H^1_!(Y,\mathcal{F}) \to H^1(Y,\mathcal{F}) \to \lim_{\to K} H^1(Y-K,\mathcal{F}).$$

Let $Y = Y_\Gamma$.

Lemma 4.5. *Let Φ_Γ be the set of cusps of Y_Γ. A cofinal system of compact K on Y is given the complements of punctured discs $\cup_{\bar{x}\in\Phi_\Gamma}\Delta^*_{\bar{x}}$ of shrinking radii (assuming regularity of cusps).*

Proof. The complement of any open subset of the form $\cup_{\bar{x}\in\Phi_\Gamma}\Delta^*_{\bar{x}}$ is closed in Y. Its preimage in $\mathcal{H}$ is closed and bounded and hence compact.

Let $K \subset Y$ be any compact subset. If we let X be a complete curve given by adding in the cusps then $K \subset X$ is a closed subset which does not contain the cusps. Thus, clearly, K is contained in the complement of some open nbhd of Φ_Γ. □

Proof of Prop 4.4. We begin by analyzing the $\lim_{\to K} H^1(Y-K,\underline{V}_k(F))$. By the previous lemma, we can restrict to K such that $Y-K$ is of the form $\cup_{\bar{x}\in\Phi_\Gamma}\Delta^*_{\bar{x}}$. Thus,

$$H^1(Y-K,\underline{V}_k(F)) \cong \oplus_{\bar{x}\in\Phi_\Gamma} H^1(\Delta^*_{\bar{x}},\underline{V}_k(F)).$$

The cohomology of a punctured disc with coefficients in a local system is something we understand. By Example 8.3 in the Appendix, the cohomology of Δ^* is same as the group cohomology of $\mathbb{Z}$ (once you choose a generator). As we restrict to smaller and smaller discs, the monodromy of $\underline{V}_k(F)$ is not changing and thus the group cohomology is also not changing. If we choose a generating loop around the cusp for each $\bar{x}$, the limit stabilizes and we have

$$\lim_{\to K} H^1(Y-K,\underline{V}_k(F)) \cong \oplus_{\bar{x}\in\Phi_\Gamma} H^1_{\mathrm{gp}(\mathbb{Z}_{\bar{x}},V_k(F))}.$$

As we mentioned earlier, the map Sh_Γ is defined in terms of map from de Rham cohomology to Hodge cohomology. We make this precise now. For any local system $\mathcal{F}$ on Y_Γ, we have an exact sequence of abelian sheaves

$$0 \to \mathcal{F} \to \mathcal{F} \otimes \mathcal{O}_{Y_\Gamma} \to \mathcal{F} \otimes \Omega^1_{Y_\Gamma} \to 0.$$

One thinks of $\mathcal{F} \otimes \Omega^1_{Y_\Gamma}$ as diffential forms valued $\mathcal{F}$. The LES in cohomology yields

$$H^0(Y_\Gamma, \mathcal{F} \otimes \mathcal{O}_Y) \to H^0(Y_\Gamma, \mathcal{F} \otimes \Omega^1_{Y_\Gamma}) \xrightarrow{\delta} H^1(Y_\Gamma, \mathcal{F}).$$

The Shimura map is given by using $V_k(\mathbb{C})$ in place of f and first associating to f an element $\omega_f \in H^0(Y_\Gamma, V_k(\mathbb{C}) \otimes \Omega^1_{Y_\Gamma})$ and then applying δ.

If we take $e_X^\vee, e_Y^\vee$ to be a basis for $V_2(\mathbb{Q})$,then explicitly

$$\omega_f = (2\pi i)(ze_X^\vee + e_Y^\vee)^{k-2} f(z) dz$$

as a differential form on $\mathcal{H}$ valued in $V_k(\mathbb{C})$, which descends to Y_Γ.

We wish to show that when we restrict $\delta(\omega_f)$ to the cusps it vanishes. The sequence above is compatible with restricting to open sets so we can replace Y_Γ with $\cup_{\bar{x} \in \Phi_\Gamma} \Delta^*_{\bar{x}}$ above. The kernel of the map δ are the closed $V_k(\mathbb{C})$-valued differential forms.

Pick a basepoint z_0. The form ω_f is closed exactly when the $\int_{z_0}^z \omega_f$ for $z \in \Delta^*_{\bar{x}}$ is independent of path. To be absolutely precise about this, we should be careful about how we make sense of the integral over a local system. Equivalently all we have to show is that the integral of ω_f around a generator the homology it is zero.

Pick a lift of $\bar{x}$ to $\mathcal{H}$ and a element $\sigma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\sigma(x) = \infty$. By our assumption about the cusps,

$$\sigma \operatorname{Stab}_\Gamma(x) \sigma^{-1} = \begin{pmatrix} 1 & n * h_x \\ 0 & 1 \end{pmatrix}.$$

A loop in the homology of $\Delta^*_{\bar{x}}$ is given by the image of the horizontal path from $\omega^{-1}(a+ib) \to \omega^{-1}(a+ib+h_x)$, where b is taken to be sufficiently large.

The integral

$$\int_{\omega^{-1}(a+ib)}^{\omega^{-1}(a+ib+h_x)} f(z) P(z,1) dz.$$

goes to zero as b goes to ∞ because f is a cusp form. $\square$

Lemma 4.6. *Pick $z_0 \in \mathcal{H}$ wih image $\bar{z}_0$ in Y_Γ. Then, the Shimura map induces a map*

$$\alpha_\Gamma : S_k(\Gamma, \mathbb{C}) \to H^1_!(\Gamma, V_k(\mathbb{C})) \subset H^1(\Gamma, V_k(\mathbb{C}))$$

which sends f to the cohomology class of the 1-cocycle given by

$$\gamma \mapsto 2\pi i \int_{z_0}^{\gamma(z_0)} f(z)(ze_X^\vee + e_Y^\vee)^{k-2} dz \in V_k(\mathbb{C}).$$

Proof. This is in Section §3.4 of [Con]. The idea is to use the interplay between the real and complex De Rham resolution of the local system $V_k(\mathbb{C})$. The real De Rham resolution is what allows us to relate the cohomology of Y_Γ to group cohomology and compatibility with the complex De Rham resolution shows how ω_f fits in. □

The injectivity of $\mathrm{Sh}_\Gamma \oplus \overline{\mathrm{Sh}}_\Gamma$ follows from compatibility of the Petersson inner and cup products in cohomology. We should recall first how the map $\overline{\mathrm{Sh}}_\Gamma$ is defined. For this, we recall some properties of complex vector spaces with real structure which both target and source of the Shimura map have. (They are not the same real structure).

Definition 4.7. If W is a complex vector space, a *real structure* on W is a a real vector subspace $W_\mathbb{R}$ such that the natural map $W_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} \to W$ is an isomorphism. Similarly, for any subfield $F \subset \mathbb{C}$-structure on W to be a F-vector space $W_F \subset W$ such that $W_F \otimes_F \mathbb{C} \cong W$.

If W has a real structure, then any element in $w \in W$ can be written uniquely as $w_1 + iw_2$ where $w_1, w_2 \in W_\mathbb{R}$. We write $w_1 = \mathrm{Re}(w)$ and $w_2 = \mathrm{Im}(w)$.

The space $H^1(\Gamma, V_k(\mathbb{C}))$ has a natural real structure given by $H^1(\Gamma, V_k(\mathbb{R}))$. Denote $\mathrm{Re}(\mathrm{Sh}_\Gamma)$ for the real part of the Shimura map which is an $\mathbb{R}$-linear map to $H^1(\Gamma, V_k(\mathbb{R}))$.

Definition 4.8. The map $\mathrm{Sh}_\Gamma \oplus \overline{\mathrm{Sh}}_\Gamma$ is defined by

$$S_k(\Gamma, \mathbb{C}) \oplus S_k(\bar{\Gamma}, \mathbb{C}) \cong S_k(\Gamma, \mathbb{C}) \otimes_\mathbb{R} \mathbb{C} \xrightarrow{2\,\mathrm{Re}(\mathrm{Sh}_\Gamma)\otimes 1} H^1(\Gamma, V_k(\mathbb{R})) \otimes_\mathbb{R} \mathbb{C} \to H^1(\Gamma, V_k(\mathbb{C})).$$

By the above definition, the Shimura map is injective if and only if $\mathrm{Re}(\mathrm{Sh}_\Gamma)$ is injective. This is what one actually checks. The injectivity is naturally deduced from a compatibility of the Petersson inner product with cup-product on the other side.

For reference, we record what the composite map looks like in terms of Sh_Γ.

Proposition 4.9. *Let X, W be $\mathbb{C}$-vector spaces, and let $X_\mathbb{R}$ be a real structure on X. Let $h : W \to X$ be a $\mathbb{C}$-linear map such that* $\mathrm{Re}(h)$ *is $\mathbb{R}$-linear. Then, the map induced by $H = 2\,\mathrm{Re}(h) \otimes 1$ from*

$$W \oplus \overline{W} \to X$$

satisfies $H((w, 0)) = h(w)$ and $H((0, \bar{w})) = \mathrm{Re}(h(w)) - i\,\mathrm{Im}(h(w)) := \overline{h(w)}$.

Proof. This is Lemma 2.9 of Bellaiche. The isomorphism $W \otimes_\mathbb{R} \mathbb{C} \cong W \oplus \bar{W}$ is given by $w \otimes \lambda \mapsto (\lambda w, \lambda \overline{w})$ so that $w \otimes 1 \mapsto (w, \overline{w})$ and $(iw) \otimes i \mapsto (-w, \overline{w})$. We deduce then that

$$H(w, \overline{w}) = 2\,\mathrm{Re}(h(w)), H(-w, \overline{w}) = -2i\,\mathrm{Im}(h(w)).$$

The formula then follows immediately from the linearity of H. □

4.2. **Petersson Inner Product.** For $f, g \in S_k(\Gamma, \mathbb{C})$, we define the Petersson inner product by

$$(f,g) = 4\pi^2 \int_{Y_\Gamma} f(z)\overline{g(z)}y^{k-2}dxdy.$$

The pairing $(\cdot,\cdot)$ is a non-degenerate Hermitian form on $S_k(\Gamma,\mathbb{C})$, with respect to which the Hecke operators are adjoint operators.

On the other side, let $\beta : H^1_c(Y,\mathcal{F}) \to H^1(Y,\mathcal{F})$ for anly local system f. Let $\mathcal{F} \otimes \mathcal{G} \to \mathcal{H}$ be any bilinear form. There is a well-defined bilinear pairing

$$< \cdot,\cdot >: H^1_!(Y,\mathcal{F}) \times H^1_!(Y,\mathcal{G}) \to H^2_c(Y,\mathcal{H}).$$

which takes a pair (a, b) to the cup product $\widetilde{a} \cup b$, where $\widetilde{a} \in H^1_c(Y,\mathcal{F})$ is any element such that $\beta(\widetilde{a}) = a$. In particular, we have a pairing

$$H^1_!(Y,\mathcal{F}) \times H^1_!(Y,\mathcal{F}^\vee) \to H^2_c(Y,\underline{\mathbb{C}}) \cong \mathbb{C}(-1)$$

for any local system $\mathcal{F}$.

Remark 4.10. Since we will be making comparison statements, we'll want all values to sit inside $\mathbb{C}$. One defines $\mathbb{Z}(1) = \ker(\exp(\mathbb{C} \to \mathbb{C}^*))$, which is a rank 1 $\mathbb{Z}$-module embedded in the complex numbers. Similarly, we can define $\mathbb{Z}(n)$ which we think of as $(2\pi i)^n\mathbb{Z} \subset \mathbb{C}$. For any characteristic 0 field F and an embedding $F \subset \mathbb{C}$, we have a natural embedding $F(n) = \mathbb{Z}(n) \otimes F \subset \mathbb{C}$, which again we think of "as" $(2\pi i)^n F \subset \mathbb{C}$.

We are almost ready to state our compatibility result. We only need some way to identify $V_k(\mathbb{C})$ and $V_k(\mathbb{C})^\vee$. There are many options here. The existence is clear if you know about the representation theory of $\mathrm{SL}_n(\mathbb{C})$, since $\mathrm{SL}_n(\mathbb{C})$ has a unique irreducible representation of any given dimension ≥ 2 which is exactly $\mathrm{Sym}^{k-2}(\mathbb{C}^2)$. However, this would only be well-defined up to $\mathbb{C}^*$. It turns out one can write down an isomorphism of $\mathrm{SL}_2(\mathbb{Z})$-modules $V_k(\mathbb{Z})$ and $V_k(\mathbb{Z})^\vee$ where the ambiguity is only multiplication by -1.

If we choose a basis $e_X^\vee, e_Y^\vee$ so identifying $V_2(\mathbb{Z}) \cong \mathbb{Z}^2$, then one checks that the action is given by the standard action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{Z}^2$. The dual action is of course given by the tranpose inverse of that. In a quirk of SL_2, the matrix $E = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ conjugates one to the other inducing the self-duality. For $k > 2$, we take the symmetric power of this duality.

For any k even ≥ 2, we get an alternating pairing

$$< \cdot,\cdot >: H^1_!(Y_\Gamma, V_k(F)) \times H^1_!(Y, V_k(F)) \to F(-1).$$

Remark 4.11. In the context of Eichler-Shimura and modular curves, the natural duality is actually between $V_k(F)$ and $V_k(F)(-k+2)$. In Brian's unavailable book [Con], this is the perspective he takes and so his constant will be off from ours by a factor of $(2\pi i)^{-k+2}$.

Proposition 4.12. *(Petersson Inner Product) Let* $f, g \in S_k(\Gamma, \mathbb{C})$*. Then,*

$$< \mathrm{Sh}_\Gamma(f_1 + \overline{g_1}), \mathrm{Sh}_\Gamma(f_2 + \overline{g_2}) >= c_k[(f_1, g_2)_\Gamma - (f_2, g_1)_\Gamma].$$

where $c_k = \frac{2^{k-2}(-1)^{k/2}}{\pi}$*. Depending on your conventions for* Sh_Γ *and the above self-duality, there a quite a number of possibilities for* c_k *so one needs to be extra careful here.*

Proof. This is Bellaiche Prop. 2.5 or Th'm 4.3.3.1 in Brian's book, however, none of their conventions are the same as ours so I sketch a proof here.

For $f, g \in S_k(\Gamma, \mathbb{C})$, define a pairing $[f, g]_k =< \mathrm{Re}(\mathrm{Sh}_\Gamma(f)), \mathrm{Re}(\mathrm{Sh}_\Gamma(g)) >$. A calculation shows that

$$\begin{aligned} < \mathrm{Sh}_\Gamma(f_1 + \overline{g}_1), \mathrm{Sh}_\Gamma(f_2 + \overline{g}_2) >=& [f_1 + g_1, f_2 + g_2]_k - [i(g_1 - f_1), i(g_2 - f_2)]_k + \\ & ([i(g_1 - f_1), f_2 + g_2]_k + [(f_1 + g_1, i(g_2 - f_2)]_k) i \end{aligned}$$

and so we can reduce to studying $[f, g]_k$. Assume we knew that

$$[if, g]_k = [f, -ig]_k.$$

With some work, one can simplify the above expression to

$$< \mathrm{Sh}_\Gamma(f_1 + \overline{g}_1), \mathrm{Sh}_\Gamma(f_2 + \overline{g}_2) >= 2([f_1, g_2]_k + [g_1, f_2]_k) + 2i([ig_1, f_2]_k + [f_1, ig_2]_k).$$

It follows that $S_k(\Gamma, \mathbb{C})$ and $\overline{S_k(\Gamma, \mathbb{C})}$ annihilate themselves under this pairing and so we are reduced to showing that

$$< \mathrm{Sh}_\Gamma(f), \mathrm{Sh}_\Gamma(\overline{g}) >= c_k(f, g)_\Gamma.$$

Note that $< \mathrm{Sh}_\Gamma(f), \mathrm{Sh}_\Gamma(\overline{g}) >= 2([f, g]_k + [f, ig]_k \cdot i)$ by setting $f_2 = g_1 = 0$.

We claim that

$$[f, g]_k = (c_k/2) \cdot i \, \mathrm{Im}(f, g)_\Gamma.$$

This would imply both the earlier identity $[if, g]_k = [f, -ig]_k$ and complete the proof. If we let $e_X^\vee, e_Y^\vee$ be a basis for $V_2(\mathbb{C})$, then under our normalization, the $V_k(\mathbb{C})$-valued form corresponding to f is given by

$$\omega_f = (2\pi i)(ze_X^\vee + e_Y^\vee)^{k-2} f(z) dz$$

and similarly for g. Using De Rham complex for computing cohomology, the trace pairing on cohomology becomes integration over the two form. The factor of $(z-\overline{z})^k$ comes from the pairing $V_k(\mathbb{C}) \times V_k(\mathbb{C}) \to \mathbb{C}$. We have

$$\begin{aligned}
[f,g]_k &= \frac{1}{2\pi i}\int_{Y_\Gamma} \mathrm{Re}(\omega_f)\wedge \mathrm{Re}(\omega_g)\\
&= \frac{1}{2\pi i}\int_{Y_\Gamma} \frac{1}{4}[(\omega_f+\overline{\omega}_f)\wedge(\omega_g+\overline{\omega}_g)]\\
&= \frac{1}{2\pi i}\int_{Y_\Gamma} \frac{-(2\pi i)^2}{4}[(z-\overline{z})^{k-2}f\overline{g}dz\wedge d\overline{z}+(\overline{z}-z)^{k-2}\overline{f}gd\overline{z}\wedge dz]\\
&= \frac{-(2\pi i)}{4}\int_{Y_\Gamma}(2iy)^{k-2}(-2i)[f\overline{g}-\overline{f}g]dx\wedge dy\\
&= \frac{(2\pi i)(2i)^{k-1}}{4}\int_{Y_\Gamma} 2i\,\mathrm{Im}(f\overline{g})y^{k-2}dx\wedge dy\\
&= (\frac{(2\pi i)(2i)^{k-1}}{4\pi^2})(\frac{i}{2})\int_{Y_\Gamma} 4\pi^2\,\mathrm{Im}(f\overline{g})y^{k-2}dx\wedge dy\\
&= (\frac{(2\pi i)(2i)^{k-1}}{4\pi^2})(\frac{i}{2})\,\mathrm{Im}(f,g)_\Gamma.
\end{aligned}$$

□

Assuming I did everything right, our constant $c_k = \frac{2^{k-2}(i)^k}{\pi}$.

4.3. **Exploiting the Cusps.** Now, we return to the study of modular symbols. The main result will be that we have a commutative diagram $(*)$:

$$\begin{array}{ccc}
S_k(\Gamma,\mathbb{C}) & \xrightarrow{\mathrm{Sh}_\Gamma} & H^1_!(Y_\Gamma, V_k(\mathbb{C}))\\
& \searrow^{\widetilde{I}} & \uparrow \beta\\
& & \mathrm{MS}_\Gamma(V_k(\mathbb{C}))
\end{array}$$

which will be Hecke-equivariant once we have defined the Hecke operators. Furthermore, we will need to identify the kernel of β as the space of boundary modular symbols and get some control over how the Hecke operators act there.

The map from modular symbols to $H^1_!$ will be defined over $\mathbb{Q}$. Obviously, the diagram only makes sense once we have gone up to $\mathbb{C}$.

Recall that we can identify $H^1_!(Y_\Gamma, V_k(\mathbb{Q}))$ in terms of group cohomology as

$$H^1_!(Y_\Gamma, V_k(\mathbb{Q})) = \ker(H^1(\Gamma, V_k(\mathbb{Q})) \to \oplus_{\overline{x}\in\Phi_\Gamma} H^1(\Gamma_x, V_k(\mathbb{Q})).$$

It is not hard then to contruct a map from $\mathrm{MS}_\Gamma(V_k(\mathbb{Q})) \to H^1(\Gamma, V_k(\mathbb{Q}))$. Let $\Psi \in \mathrm{MS}_\Gamma(V_k(\mathbb{Q}))$. We define a 1-cocycle valued in $V_k(\mathbb{Q})$ by

$$\gamma \mapsto \Psi\infty \to \gamma(\infty) \in V_k(\mathbb{Q}).$$

It is easy to check that the Γ-equivariance of $\Psi \in \mathrm{Hom}(\Delta_0, V_k(\mathbb{Q}))$ makes this into a co-cycle. We denote the cohomology class by $[\Psi] = \beta(\Psi)$.

The condition that Ψ lands in $H^1_!$ is that $\mathrm{Res}_{\Gamma_x}([\Psi]) = 0$, where $\Gamma_x \subset \Gamma$ is the stabilizer of any cusp $x \in \mathbb{P}^1(\mathbb{Q})$.

Proposition 4.13. *The restriction of* $[\Psi]$ *to* Γ_x *is trivial.*

Proof. If $x = \infty$ and $\sigma \in \Gamma_\infty$, then by the way we defined $[\Psi]$, we have $[\Psi]_\infty = \Psi\{\infty \to \infty\} = 0$.

Assume $x \neq \infty$ and $\sigma \in \Gamma_x$. Then, we have

$$\begin{aligned} \Psi\{\infty \to \sigma(\infty)\} &= \Psi\{\infty \to x\} + \Psi\{x \to \sigma(\infty)\} \\ &= \Psi\{\infty \to x\} + \Psi\{\sigma(x) \to \sigma(\infty)\} \\ &= \Psi\{\infty \to x\} + \sigma.\Psi\{x \to \infty\} \\ &= \sigma.\Psi\{x \to \infty\} - \Psi\{x \to \infty\}. \end{aligned}$$

This realizes $\mathrm{Res}_{\Gamma_x}([\Psi])$ as the coboundary given by $\Psi\{x \to \infty\} \in V_k(\mathbb{Q})$. □

That the diagram commutes can almost be seen from Lemma 4.6. In that Lemma, however, we chose a basepoint $z_0 \in Y$. We would like to drag that basepoint out to the cusp ∞.

Proposition 4.14. *The diagram* $(*)$ *commutes.*

Proof. Let $f \in S_k(\Gamma, \mathbb{C})$. Pick a basepoint $z_0 \in \mathcal{H}$. By Lemma 4.6, we can identify the cohomology class of $\mathrm{Sh}_\Gamma(f)$, as being the cohomology class of the co-cycle

$$(I_f^0)_\gamma(P) := \int_{z_0}^{\gamma(z_0)} f(z)P(z,1)dz.$$

Whereas the cohomology class given by passing through modular symbols is given by

$$(I_f)_\gamma(P) := \int_{\infty}^{\gamma(\infty)} f(z)P(z,1)dz.$$

The claim is that $(I_f^0)_\gamma$ and $(I_f)_\gamma$ represent the same cohomology class. In some sense, there is really no real work being done here. The difficult part was knowing (using that f is a cusp form) that we could integrate from one cusp to another. Given that all the integrals converge, the following computation identifies the co-boundary which relates the two

$$\int_{z_0}^{\gamma(z_0)} f(z)P(z,1)dz - \int_{\infty}^{\gamma(\infty)} f(z)P(z,1)dz = \int_{z_0}^{\infty)} f(z)P(z,1)dz + \int_{\gamma(\infty)}^{\gamma(z_0)} f(z)P(z,1)dz$$
$$= \int_{z_0}^{\infty)} f(z)P(z,1)dz - \int_{z_0}^{\infty} f(z)P_{|_\gamma}(z,1)dz.$$

□

Corollary 4.15. *The map* $\widetilde{I} : S_k(\Gamma, \mathbb{C}) \to \mathrm{MS}_\Gamma(V_k(\mathbb{C}))$ *is injective. In fact, the induced map* $\widetilde{I} \oplus \overline{\widetilde{I}} : S_k(\Gamma, \mathbb{C}) \oplus \overline{S_k(\Gamma, \mathbb{C})} \to \mathrm{MS}_\Gamma(V_k(\mathbb{C}))$ *is injective and maps isomorphically to the quotient* $\mathrm{MS}_\Gamma(V_k(\mathbb{C}))/\ker(\beta_\mathbb{C})$.

Next, we would like to study the kernel of $\beta_k : \mathrm{MS}_\Gamma(V_k(\mathbb{Q})) \to H^1_!(Y_\Gamma, V_k(\mathbb{Q})$ which we denote $\mathrm{BMS}_\Gamma(V_k(\mathbb{Q}))$. I should mention that in Bellaiche's notes [Bel] he contructs a isomorphism "directly" between the space of modular symbols and the compact cohomology of Y_Γ. We will essentially do the same by putting $\mathrm{MS}_\Gamma(V_k(\mathbb{Q})$ into the same exact sequence we used in the proof of Prop. 4.4 in place of H^1_c.

The kernel is not so hard to identify:

$$\mathrm{BMS}_\Gamma(V_k(\mathbb{Q})) = \{\Psi \in \mathrm{Hom}_\Gamma(\Delta_0, V_k(\mathbb{Q}))\}$$

such that $\Psi\{\infty \to \gamma(\infty)\} = \gamma(v) - v$ for some $v \in V_k(\mathbb{Q})$. That is Ψ defines a coboundary.

Proposition 4.16. *Let* Φ_Γ *be a choice of representatives in* $\mathbb{P}^1(\mathbb{Q})$ *for the set of cusps of* Y_Γ *(= the* Γ*-orbits on* $\mathbb{P}^1(\mathbb{Q})$*). For* $x \in \Phi_\Gamma$*, let* Γ_x *denote the stabilizer of* x *in* Γ*. For any* Γ*-module* M*, there exists an exact sequence*

$$0 \to M^\Gamma \to \oplus_{x\in\Phi_\Gamma} M^{\Gamma_x} \to \mathrm{BMS}_\Gamma(M) \to 0$$

Before we prove the Proposition, we note the similarity with the exact sequence

$$0 \to H^0(Y_\Gamma, \underline{M}) \to \oplus_{x\in\Phi_\Gamma} H^0(\Delta^*_{\overline{x}}, \underline{M}) \to H^1_c(Y_\Gamma, \underline{M}) \to H^1_!(Y_\Gamma, \underline{M}) \to 0.$$

We remove the $\lim_K$ because identifying $H^0(\Delta^*_{\overline{x}}, \underline{M}) = M^{\Gamma_x}$, it is clear the the limit is stable as the radii shrink.

Proof of Prop 4.16. Label the elements in Φ_Γ as $x_1, \ldots, x_n$. Let $\nu = (v_1, \ldots, v_n) \in \oplus_{x\in\Phi_\Gamma} M^{\Gamma_x}$. We define a modular symbol Ψ_ν as follows.

We actually define an element in $\mathrm{Hom}_\Gamma(\Delta, M)$ which we can then restrict to Δ_0. For any $a \in \mathbb{P}^1(\mathbb{Q})$, a lies in the orbit of some x_i. We have an element $\gamma \in \Gamma$ such that $\gamma(x_i) = a$ which is well-defined up to right multiplication by an element in Γ_{x_i}. Define

$$\Psi_\nu(\{a\}) := \gamma.v_i.$$

This is independent of our choice of γ because v_i is invariant by Γ_{x_i}. The Γ-invariance of the map is apparent. It is also clear that Ψ_ν defines a boundary modular symbol since

$$\Psi_\nu(\{\infty - \gamma(\infty)\}) = \gamma_\infty(v_i) - \gamma.(\gamma_\infty(v_i)),$$

where ∞ is in the orbit of x_i with $\gamma_\infty(x_i) = \infty$.

Next, we check the kernel. We have

$$\Psi_\nu\{x_i \to \gamma.x_i\} = 0 \Longrightarrow v_i \in M^\Gamma$$

and

$$\Psi_\nu\{x_i \to x_j\} = 0 \Longrightarrow v_i = v_j$$

so the kernel is exactly M^Γ.

It remains to check surjectivity. Let $\Psi \in \mathrm{BMS}_\Gamma(M)$. By definition that means, $\Psi\{\infty \to \gamma(\infty)\} = v_0 - \gamma.v_0$ for some $v_0 \in M$. Observe that v_0 must be in M^{Γ_∞}. Assume WLOG that ∞ is in the Γ-orbit of x_1 and choose σ such that $\infty = \sigma(x_1)$. We set $v_1 = \sigma^{-1}.v_0$ which one can check is invariant under $\Gamma_{x_1} = \sigma^{-1}\Gamma_\infty\sigma$.

Knowing v_1, we have a candidate for ν namely

$$v_i := \Psi\{x_i \to x_1\} + v_1.$$

I leave it to the reader to check that $v_i \in M^{\Gamma_{x_i}}$. We verify that $\Psi_\nu = \Psi$. It is not hard to check that

$$\Psi\{\infty \to \gamma(\infty)\} = \sigma(v_1) - \gamma(\sigma(v_1)) = \Psi_\nu\{\infty \to \gamma(\infty)\}.$$

Since they agree on the ∞-orbit, we have that $\Psi\{\gamma(x_1) - \gamma'(x_1)\} = \gamma.v_1 - \gamma'.v_2$. Then, we proceed with the following calculation:

$$\begin{aligned}
\Psi_\nu\{a \to b\} &= \gamma_a(v_i) - \gamma_b(v_j) \\
&= \gamma_a.(\Psi\{x_i \to x_1\}) + \gamma_a.v_1 - \gamma_b.(\Psi\{x_j \to x_1\}) - \gamma_b.v_1 \\
&= \Psi\{a \to \gamma_a(x_1)\} + \gamma_a.v_1 + \Psi\{\gamma_b(x_1) \to b\} - \gamma_b.v_1 \\
&= \Psi\{a \to \gamma_a(x_1)\} + \Psi\{\gamma_a(x_1) - \gamma_b(x_1)\} + \Psi\{\gamma_b(x_1) \to b\} \\
&= \Psi\{a \to b\}
\end{aligned}$$

This last argument is just tracing along a quadrilateral connecting a, b to the orbit of x_1. □

Corollary 4.17. *We have the following four-term exact sequence with $F = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$:*

$$0 \to \mathrm{BMS}_\Gamma(V_k(F)) \to \mathrm{MS}_\Gamma(V_k(F)) \to H^1(\Gamma, V_k(F)) \to E_k(\Gamma, F) \to 0$$

where $E_k(\Gamma, F)$ is the cokernel fo $H^1_! \to H^1$ (it is the Eisenstein part of the cohomology). There exists a perfect pairing

$$\mathrm{BMS}_\Gamma(V_k(F)) \times E_k(F) \to F(-1)$$

for which Hecke operators is "self-adjoint" meaning $< T_\ell(\Psi), g >=< \Psi, T_\ell(g) >$ (once this has been defined).

Proof. This is Th'm 2.15 in Bellaiche notes. A sketch of a proof is as follows. Poincare duality and the self-duality of $V_k(F)$ gives us a perfect pairing

$$H^1(Y_\Gamma, V_k(F)) \times H^1_c(Y_\Gamma, V_k(F)) \to F(-1).$$

This is compatible with the pairing $H^1_c \times H^1_c \to F(-1)$. The exact sequence in Prop 4.16 allows us to identify $\mathrm{BMS}_\Gamma(V_k(F))$ with $\ker(\beta) : H^1_c \to H^1$. The compatibility of the two pairing shows that $< \ker(\beta), H^1_! >= 0$ is perpendicular to the $H^1_! \subset H^1$. The fact the pairing is perfect is equivalent to the perfectness of the pairing $H^1_! \times H^1_!$.

Let $x_1 \in H^1_!$, then $x_1 = \beta(y_1)$ for some $y_1 \in H^1_c$. For any $x_2 \in H^1_!$, the pairing is given by pairing $< y_1, x_2 >$, where x_2 is thought of as an element of H^1. There exists some $z \in H^1$ such that $< y_1, z > \neq 0$.

The Hecke equivariance follows from the Hecke equivariance of the pairing on $H^1_c \times H^1$.The fact that $E_k(\Gamma, F)$ really is spanned by Eisenstein series is a non-trivial but classical fact in the study of modular forms (ADD REFERENCE). □

5. Hecke Operators

From now on, we would like to work with modular forms of level $\Gamma_0(N)$. However, in the previous sections, we assumed that Γ was acting freely with regular cusps. This may not be true for $\Gamma_0(N)$. We may however choose a smaller discrete normal subgroup $\Gamma \subset \Gamma_0(N)$, and then we would like to identify $S_k(\Gamma_0(N), \mathbb{C}) \subset S_k(\Gamma, \mathbb{C})$ and that we still have what we need for this larger group.

Basically, we have

$$\widetilde{I} : S_k(\Gamma, \mathbb{C}) \to \mathrm{MS}_\Gamma(V_k(\mathbb{C})).$$

We will contruct an action of $\Gamma_0(N)$ on both sides such that $\widetilde{I}$ is a $\Gamma_0(N)$-equivariant map and such that the invariance will be exactly $S_k(\Gamma_0(N), \mathbb{C})$ and $\mathrm{MS}(\Gamma_0(N), V_k(\mathbb{C}))$ respectively.

Recall the notation for the slash operator on modular forms. If $A \in \mathrm{GL}_2(\mathbb{Q})$ with $\det(A) > 0$, then we define $\rho(A) = \frac{\det(A)^{1/2}}{cz+d}$. The origin of this factor is that

$$d(A(z)) = \rho(A)^{-2} dz$$

and if $A \in \mathrm{SL}_2(\mathbb{Z})$ then $\rho(A)^{-k}$ is what shows up in the condition for being a modular.

We define

$$f|_A(z) := \rho(A)^k f(A(z))$$

for any $A \in \mathrm{GL}_2(\mathbb{Q})$ with $\det(A) > 0$, $f \in S_k(\Gamma, \mathbb{C})$. One checks that if we restrict to $\Gamma_0(N)$, this defines a right action of $\Gamma_0(N)$ on $S_k(\Gamma, \mathbb{C})$. Furthermore,

$$S_k(\Gamma, \mathbb{C})^{\Gamma_0(N)} = S_k(\Gamma_0(N), \mathbb{C}).$$

Lemma 5.1. *Let $f \in S_k(\Gamma, \mathbb{C})$, $P \in \mathcal{P}_k(\mathbb{C}), A \in \mathrm{GL}_2(\mathbb{Q})^+$. Then,*

$$f(A(z))P(A(z), 1)d(A(z)) = f|_A(z)P|_A(z, 1)dz.$$

Proof. I leave this as an exercise as a reader in keeping track of powers of $\rho(A)$. I should point out that this identity was essentially used in Prop 3.7. Also, the slash operator as defined on $\mathcal{P}_k(\mathbb{C})$ earlier in §3 exactly has the property that $P|_A(z, 1) = \rho(A)^{2-k} P(A(z), 1)$. □

Definition 5.2. Let $\Psi \in \mathrm{MS}_\Gamma(V_k(F))$. We define a right action of $\mathrm{GL}_2(\mathbb{Q})^+$ by

$$\Psi|_A\{r \to s\}(P) := \Psi\{A(r) \to A(s)\}(P|_{A^{-1}}).$$

It is not hard to see that under this definition the map I is equivariant for the right $\mathrm{GL}_2(\mathbb{Q})^+$-action. This crucially uses Lemma 5.1.

Proposition 5.3. *Consider the right action of $\Gamma_0(N)$ on $\mathrm{MS}_\Gamma(V_k(F))$, where $\Gamma \subset \Gamma_0(N)$. Then,*

$$\mathrm{MS}_\Gamma(V_k(F))^{\Gamma_0(N)} = \mathrm{MS}_{\Gamma_0(N)}(V_k(F)).$$

Proof. Recall that the right-hand side is defined to be $\Gamma_0(N)$-equivariant maps from Δ_0 to $V_k(F)$. Concretely, for $\Psi \in MS_\Gamma(V_k(F))$, this is the condition that

$$\Psi\{\gamma(r) \to \gamma(s)\}(P) = \Psi\{r \to s\}(P|_\gamma)$$

for all $\gamma \in \Gamma_0(N)$. Making the substitution, $P \mapsto P|_{\gamma^{-1}}$, we get the equivalent condition

$$\Psi\{\gamma(r) \to \gamma(s)\}(P|_{\gamma^{-1}}) = \Psi\{r \to s\}(P),$$

which is the same thing as being fixed under the right $\Gamma_0(N)$-action as defined above. □

We continue to use $\widetilde{I}$ to denote the map $S_k(\Gamma_0(N),\mathbb{C}) \to \mathrm{MS}_{\Gamma_0(N)}(V_k(F))$ which we now see is compatible with shrinking the congruence subgroup Γ. As for the group Γ, we have a diagram

$$\begin{array}{ccc} S_k(\Gamma_0(N),\mathbb{C}) & \xrightarrow{\mathrm{Sh}_\Gamma} & H^1_!(Y_\Gamma, V_k(\mathbb{C}))^{\Gamma_0(N)} \\ & \searrow^{\widetilde{I}} & \uparrow \beta \\ & & \mathrm{MS}_{Gamma_0(N)}(V_k(\mathbb{C})). \end{array}$$

The only difference being that it is not clear if the right-hand side can be interpreted as a cohomology group because $\Gamma_0(N)$ may not act freely on the upper-half plane. Other than that we have all the same properties that we had before Γ. In particular, we have an exact sequence

$$0 \to \mathrm{BMS}_{\Gamma_0(N)}(V_k(F)) \to \mathrm{MS}_{\Gamma_0(N)}(V_k(F)) \to H^1(\Gamma, V_k(F))^{\Gamma_0(N)} \to E_k(\Gamma, F)^{\Gamma_0(N)} \to 0$$

Assume Γ is normal in $\Gamma_0(N)$. Since F is characteristic 0, taking invariance under the finite group $\Gamma_0(N)/\Gamma$ is exact. By the inflation restriction early term exact sequence,

$$H^1(\Gamma, V_k(F))^{\Gamma_0(N)} \cong H^1(\Gamma_0(N), V_k(F)),$$

since finite groups have no higher cohomology in characteristic 0. We define $E_k(\Gamma, F)^{\Gamma_0(N)} =: E_k(\Gamma_0(N), F)$.

We record the following fact which must be deduced from the classical theory:

Proposition 5.4. *The space $E_k(\Gamma_0(N),\mathbb{C})$ is spanned by the Eisenstein series of weight k and level $\Gamma_0(N)$ compatible with the Hecke action. By spanned, I mean that we take both f and $\overline{f}$ under the corresponding Shimura map for Eisenstein series.*

Proof. (ADD REFERENCE) □

Now, finally, we recall the classical Hecke operators on modular forms of level $\Gamma_0(N)$. Theoretically, it is probably better to formulate them as double coset operators as this formulation would make clearer how they act on group cohomology and their behavior with respect to restricting to smaller subgroups which is useful in the discussion above. However, we opt for the more concrete approach since we will need to prove various formulas using them. It also makes it clearer how they should act on modular symbols.

For each prime ℓ not dividing N and each $f \in S_k(\Gamma_0(N),\mathbb{C})$, we define the ℓth Hecke operator by

$$T_\ell(f) = \ell^{k/2-1}\left(\sum_{a=0}^{\ell-1} f|\begin{bmatrix} 1 & a \\ 0 & \ell \end{bmatrix} + f|\begin{bmatrix} \ell & 0 \\ 0 & 1 \end{bmatrix}\right).$$

For $p \mid N$, we define the Hecke operator U_p, by

$$U_p(f) = p^{k/2-1}\left(\sum_{a=0}^{p-1} f|_{\left[\begin{smallmatrix}1 & a\\ 0 & p\end{smallmatrix}\right]}\right).$$

THe Hecke operators commute with each other and so we can form the commutative algebra $\mathbb{T} = \mathbb{Z}[T_\ell, U_p]$. A Hecke eigenform f defines a character $\lambda_f : \mathbb{T} \to \mathbb{C}^*$. Recall that this has the nice property that if f is normalized then $T_\ell(f) = a_\ell f$ so that λ_f takes values in the coefficient field of f.

It is clear then how to define the Hecke operators on the space of modular symbols.

Definition 5.5. Let $\Psi \in \mathrm{MS}_\Gamma(V_k(\mathbb{Q}))$. If ℓ does not divide N, then we define

$$T_\ell(\Psi) := \ell^{k/2-1}\left(\sum_{a=0}^{\ell-1} \Psi|_{\left[\begin{smallmatrix}1 & a\\ 0 & \ell\end{smallmatrix}\right]} + \Psi|_{\left[\begin{smallmatrix}\ell & 0\\ 0 & 1\end{smallmatrix}\right]}\right).$$

If $\ell \mid N$, then we define For $p \mid N$, we define the Hecke operator U_p, by

$$U_p(\Psi) = p^{k/2-1}\left(\sum_{a=0}^{p-1} \Psi|_{\left[\begin{smallmatrix}1 & a\\ 0 & p\end{smallmatrix}\right]}\right).$$

Proposition 5.6. *The Hecke operators T_ℓ, U_p preserve the space $\mathrm{MS}_{\Gamma_0(N)}(V_k(\mathbb{Q}))$. The map $\widetilde{I}$: $S_k(\Gamma_0(N), \mathbb{C}) \to \mathrm{MS}_{\Gamma_0(N)}(V_k(\mathbb{C}))$ is Hecke-equivariant.*

Proof. The first part follows from the same argument one uses to show that Hecke operators preserve the space of modular forms. The second part follows from the $\mathrm{GL}_2(\mathbb{Q})^+$ equivariance of the map $\widetilde{I}$. □

Lemma 5.7. *For all $p \mid N$, $\mathrm{MS}_{\Gamma_0(N)}(V_k(\mathbb{Q}))$, we have*

$$U_p(\Psi)\{r \to s\}(P) = \sum_{a=0}^{p-1} \Psi\{\gamma_a(r) \to \gamma_a(s)\}(P|_{p\gamma_a^{-1}}),$$

where $\gamma_a = \left[\begin{smallmatrix}1 & a\\ 0 & p\end{smallmatrix}\right]$.

Proof. This is Lemma 1.2 in [BDInv]. Recall that in that paper the slash operator is defined without the determinant condition. Nevertheless, the formula remains the same because $p\gamma_a^{-1} \in \mathrm{SL}_2(\mathbb{Q})$. The formula easily reduces to the following equality

$$p^{\frac{k-2}{2}} P|_{\gamma_a^{-1}} = P|_{p\gamma_a^{-1}}.$$

We can pull the p out of the homogeneous polynomial so that

$$P|_{p\gamma_a^{-1}} = p^{k-2} \cdot \det(\gamma_a^{-1})^{\frac{k-2}{2}} \cdot P|_{\gamma_a^{-1}}$$

and $\det(\gamma_a^{-1}) = p^{-1}$, and so we everything works out. □

6. Removing Complex Conjugation

To a modular form f, we would like to associate a one-dimensional subspace of modular symbols. As in the case of the Shimura map, what is most naturally associated to f is the two-dimensional space spanned by $f \oplus \overline{f}$. To fix this problem, we introduce a Hecke operator "at infinity," which will break up the space of modular symbols into plus and minus parts. This will allow us to define the plus and minus periods of a modular form f and lead to the main result of this talk.

In our example, we noticed that the period of $f(z) = q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{11n})^2$ from $\{0 \to \infty\}$ was purely real. Let's see more generally when this happens. Let's work in the weight 2 case for a moment.

If $f \in S_k(\Gamma, \mathbb{R})$, then it has real Fourier coefficients. This gives f the following symmetry

$$f(-\overline{z}) = \overline{f(z)}$$

because the same holds for $q(z) = e^{2\pi i z}$.

Note that $z \mapsto -\overline{z}$ is an involution of the upper half plane. (If we were doing this properly over $\mathbb{C} - \mathbb{R}$, we wouldn't need the minus sign.) In any case, we get

$$\overline{2\pi i \int_0^\infty f(z)dz} = (-2\pi i)\int_0^\infty \overline{f(z)}d(\overline{z}) = (-2\pi i)\int_0^\infty f(-\overline{z})d(\overline{z}).$$

If we make the substitution $z \mapsto -\overline{z}$, the path $0 \to \infty$ is preserved so we get

$$\overline{2\pi i \int_0^\infty f(z)dz} = (-2\pi i)\int_0^\infty f(z)d(-z)$$

and the two minus signs cancel so the value of period is real.

On the modular symbols side, the involution "corresponding" to $z \mapsto -\overline{z}$ is given by

$$c = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right].$$

We define an involution on the space of the space of modular symbols by

$$\iota(\Psi) = \Psi|_c.$$

Since $\det(c) < 0$, I should be clear that what I mean is $\iota(\Psi)\{r \to s\}(P(X,Y)) := \Psi\{-r \to -s\}(P(-X,Y)$. Because c normalizes the $\Gamma_0(N)$, the involution ι is easily seen to act on the space $\mathrm{MS}_{\Gamma_0(N)}(V_k(F))$.

Proposition 6.1. *The involution ι commutes with all Hecke operators T_ℓ, U_p.*

Proof. For any given T_ℓ or U_p, we can pass $|_c$ through the sum. To show that $\iota T_\ell \iota = T_\ell$ and $\iota U_p \iota = U_p$, one first notes that

$$c\left[\begin{smallmatrix} \ell & 0 \\ 0 & 1 \end{smallmatrix}\right]c = \left[\begin{smallmatrix} \ell & 0 \\ 0 & 1 \end{smallmatrix}\right]$$

and

$$c\left[\begin{smallmatrix} 1 & a \\ 0 & \ell \end{smallmatrix}\right]c = \left[\begin{smallmatrix} 1 & -a \\ 0 & \ell \end{smallmatrix}\right]$$

Since we are working modulo the action of $\Gamma_0(N)$ which contains $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$, the operator $|_{\left[\begin{smallmatrix} 1 & -a \\ 0 & \ell \end{smallmatrix}\right]} = |_{\left[\begin{smallmatrix} 1 & -a+\ell \\ 0 & \ell \end{smallmatrix}\right]}$. Basically, conjugation by ι permutes the double coset representatives corresponding to T_ℓ or U_p respectively. □

We write $\mathrm{MS}^{\pm}_{\Gamma_0(N)}(V_k(F))$ for the $\pm$-eigenspace of ι. These are Hecke-stable subspaces. Furthermore, any modular symbol Ψ can be written uniquely as

$$\Psi = \Psi^+ + \Psi^-,$$

where $\Psi^{\pm}$ lies in $MS^{\pm}_{\Gamma_0(N)}(V_k(F))$. In particular, for any $f \in S_k(\Gamma_0(N), \mathbb{C})$, we can write

$$\widetilde{I}_f = \widetilde{I}_f^+ + \widetilde{I}_f^-.$$

Now, we are ready to state our main result:

Theorem 6.2. *(Prop 1.1. [BDInv]) Let $f \in S_k(\Gamma_0(N), \mathbb{C})$ be a normalized Hecke eigenform, with coefficient field K_f. Choose an embedding of $K_f \to \mathbb{C}$. There exists complex periods Γ_f^+ and Γ_f^- with the property that*

$$I_f^+ := (\Omega_f^+)^{-1}\widetilde{I}_f^+, \quad I_f^- := (\Omega_f^-)^{-1}\widetilde{I}_f^-$$

belong to $\mathrm{MS}_{\Gamma_0(N)}(V_k(K_f))$. *These periods can be chosen to satisfy*

$$\Omega_f^+\Omega_f^- =< f, f >_{\Gamma_0(N)} .$$

This rationality statement is absolutely essential for moving the modular symbol associated to f from $\mathbb{C}$ to $\overline{\mathbb{Q}_p}$, and thus to begin our study of the p-adic properties of periods. And hence, why we spent so much time setting it up, despite its relatively innocuous looking form.

The rough idea behind the proof is that one can put a canonical K_f structure on the one-dimensional space corresponding to $\widetilde{I}_f^+$ and $\widetilde{I}_f^-$ respectively. This one-dimensional space is a particular eigenspace for the Hecke operators. The formula for the product follows from a computation of how the pairing behaves with respect to $+, -$ spaces using Prop 4.12.

Lemma 6.3. *Consider the map* $\widetilde{I} \oplus \overline{\widetilde{I}} : S_k(\Gamma, \mathbb{C}) \oplus \overline{S_k(\Gamma, \mathbb{C})} \to \mathrm{MS}_\Gamma(V_k(\mathbb{C}))$*, defined in the same way as in* 4.8*. Then,*

$$\iota(\widetilde{I}_f) = \overline{\widetilde{I}}(\bar{f}),$$

for $f \in S_k(\Gamma, \mathbb{R})$*.*

Proof. The key calculation is very similar to one we did earlier. Namely, let $f \in S_k(\Gamma, \mathbb{R})$ and $P \in \mathcal{P}_k(\mathbb{R})$, then

$$\begin{aligned} \iota(\widetilde{I}_f)\{r \to s\}(P) &= 2\pi i \int_{-r}^{-s} f(z)P(-z,1)dz \\ &= -2\pi i \int_r^s f(-\overline{z})P(\overline{z},1)d\overline{z} \\ &= -2\pi i \int_r^s \overline{f(z)P(z,1)}d\overline{z} \\ &= \overline{2\pi i \int_r^s f(z)P(z,1)dz}. \end{aligned}$$

This calculation says that for $f \in S_k(\Gamma, \mathbb{R})$ and $P \in \mathcal{P}_k(\mathbb{R})$, $\iota(\widetilde{I}_f) = \mathrm{Re}(\widetilde{I}_f) - i\,\mathrm{Im}(\widetilde{I}_f)$. By Prop. 4.9, this is exactly the map $\overline{\widetilde{I}}(\bar{f})$. □

Proposition 6.4. *The involution* ι *induces an involution on the boundary modular symbols* $\mathrm{BMS}_\Gamma(V_k(F))$*.*

Proof. We just need to check that ι preserves the kernel of the morphism $\beta : \mathrm{MS}_\Gamma(V_k(F)) \to H^1_!(\Gamma, V_k(F))$. A modular symbol Ψ is in the kernel if for all γ, $\Psi\{\infty \to \gamma(\infty)\} = \gamma(v) - v$ for some fixed $v \in V_k(F)$. This is somewhat tedious and unenlightening computation which I leave to the reader. A hint is that the coboundary for $\Psi|_c$ is given by v^c, where $v^c \in V_k(F)$ is the functional $v^c(P) = v(P(-X,Y))$. □

Theorem 6.5. *The map* $f \mapsto \widetilde{I}^{\pm}_f$ *is a* $\mathbb{C}$*-linear Hecke compatible map* $S_k(\Gamma_0(N), \mathbb{C}) \to \mathrm{MS}^{\pm}_{\Gamma_0(N)}(V_k(\mathbb{C}))$ *such that the induced map*

$$S_k(\Gamma_0(N), \mathbb{C}) \oplus \mathrm{BMS}^{\pm}_{\Gamma_0(N)}(V_k(\mathbb{C})) \to \mathrm{MS}^{\pm}_{\Gamma_0(N)}(V_k(\mathbb{C}))$$

is a Hecke-compatible isomorphism.

Proof. This is Th'm 2.14 in Bellaiche. By Eichler Shimura (Th'm 4.3) and staring at the diagram $(*)$, we see that

$$2\dim_{\mathbb{C}} S_k(\Gamma_0(N), \mathbb{C}) = \dim_{\mathbb{C}} \mathrm{MS}_{\Gamma_0(N)}(V_k(\mathbb{C})) - \dim_{\mathbb{C}} \mathrm{BMS}_{\Gamma_0(N)}(V_k(\mathbb{C})).$$

If we can show that both $\widetilde{I}^{+}$ and $\widetilde{I}^{-}$ are injective, then their images must fill up the entire complement of $\mathrm{BMS}_{\Gamma_0(N)}(V_k(\mathbb{C}))$ and the non-boundary part of $\mathrm{MS}^{\pm}_{\Gamma_0(N)}(V_k(\mathbb{C}))$ must have the same dimension. The isomorphism above would then follow.

We can check injectivity after composition with β, and furthermore, we can restrict to the real subspace $S_k(\Gamma_0(N),\mathbb{R})$. For $f \in S_k(\Gamma_0(N),\mathbb{R})$, Lemma 6.3 shows that

$$\widetilde{I}^{\pm}_f = \frac{1}{2}(\widetilde{I}_f \pm \iota(\widetilde{I}_f)) = \frac{1}{2}(\widetilde{I}_f \pm \overline{\widetilde{I}}(\overline{f})).$$

Composing with β, we get the map $f \mapsto \frac{1}{2}\,\mathrm{Sh}_{\Gamma_0(N)}(f,\overline{f})$ for the plus part and $f \mapsto \frac{1}{2}\,\mathrm{Sh}_{\Gamma_0(N)}(f,-\overline{f})$. The injectivity of $\mathrm{Sh}_{\Gamma_0(N)}$ implies injectivity of $\widetilde{I}^{\pm}_f$. □

Lemma 6.6. *Let $f \in S_k(\Gamma_0(N),\mathbb{C})$. Then,*

$$< \widetilde{I}^{+}_f, \widetilde{I}^{-}_f >= 1/2 c_k (f,f)_{\Gamma_0(N)}.$$

Recall that $c_k = \frac{2^{k-2}(-1)^{k/2}}{\pi}$.

Proof. Both sides behave same with respect to $\mathbb{C}$-scaling and taking sums of forms $f+g$. Thus, it suffices to consider $f \in S_k(\Gamma_0(N),\mathbb{R})$. Writing $\widetilde{I}^{\pm}_f = \frac{1}{2}(\widetilde{I}_f \pm \iota(\widetilde{I}_f))$ and using that $<\cdot,\cdot>$ is an alternating pairing, we get that

$$< \widetilde{I}^{+}_f, \widetilde{I}^{-}_f >= 1/2 < \widetilde{I}_f, \iota(\widetilde{I}_f) > .$$

And applying Lemma 6.3, we have

$$< \widetilde{I}^{+}_f, \widetilde{I}^{-}_f >= 1/2 < \widetilde{I}_f, \overline{\widetilde{I}}_{\overline{f}} > .$$

Compatiblity with the Shimura map and Prop 4.12, shows that

$$< \widetilde{I}^{+}_f, \widetilde{I}^{-}_f >= 1/2 < \mathrm{Sh}_{\Gamma_0(N)}(f), \mathrm{Sh}_{\Gamma_0(N)}(\overline{f}) >= 1/2 \cdot c_k \cdot (f,f)_{\Gamma_0(N)}.$$

□

Proof of Th'm 6.2. Since we have the isomorphism above for all N, we can assume $f \in S_k(\Gamma_0(N),\mathbb{C})$ is a normalized newform (does not come from lower level), which is a Hecke eigenform. This is not such an important point, but to be sure what I say is correct, we should also restrict to the newspaces both on the modular forms and the modular symbols side.

Let $\mathbb{T}$ denote the Hecke-algebra generated by the T_ℓ and U_p. Since f is an eigenform, it defines a character $\chi_f : \mathbb{T} \to K_f$ (under our chosen embedding of $K_f \to \mathbb{C}$). Since $\widetilde{I}^{\pm}$ is Hecke compatible, we have

$$S_k(\Gamma_0(N),\mathbb{C})[\chi_f] \hookrightarrow \mathrm{MS}^{\pm}_{\Gamma_0(N)}(V_k(\mathbb{C}))[\chi_f].$$

I claim that the right-hand side is always 1-dimensional. By Th'm 6.5, it suffices to show that $\mathrm{BMS}_{\Gamma_0(N)}(V_k(\mathbb{C}))[\chi_f] = \emptyset$. By Cor 4.17, the space $\mathrm{BMS}_{\Gamma_0(N)}(V_k(\mathbb{C}))$ is dual to the space of Eisenstein cohomology. The content of the result then is that χ_f is not a Hecke-character associated to an Eisenstein series of weight k and level $\Gamma_0(N)$. (EXPLAIN MORE)

Because χ_f takes values in the subfield $K_f \hookrightarrow \mathbb{C}$ and all the Hecke operators are in fact defined over $\mathbb{Q}$, we see that the 1-dimensional $\mathbb{C}$-space $\mathrm{MS}^{\pm}_{\Gamma_0(N)}(V_k(\mathbb{C}))[\chi_f]$ has a natural K_f structure given by $\mathrm{MS}^{\pm}_{\Gamma_0(N)}(V_k(K_f))[\chi_f]$. We can write

$$\widetilde{I}_f^{\pm} = \Omega_f^{\pm} I_f^{\pm},$$

where $I_f^{\pm} \in \mathrm{MS}^{\pm}_{\Gamma_0(N)}(V_k(K_f))[\chi_f]$ and $\Omega_f^{\pm}$ is a complex number well-defined up to multiplication by $K_f^{\times}$.

If we can show that

$$\frac{(f,f)_\Gamma}{\Omega_f^+\Omega_f^-} \in K_f^{\times},$$

then we can adjust Ω_f^+ so that $(f,f)_\Gamma = \Omega_f^+\Omega_f^-$. Consider then that

$$\frac{< \widetilde{I}_f^+, \widetilde{I}_f^- >}{\Omega_f^+\Omega_f^-} =< I_f^+, I_f^- >\in K_f(-1) = \frac{1}{2\pi i}K_f$$

so that $(2\pi i) * \frac{<\widetilde{I}_f^+,\widetilde{I}_f^->}{\Omega_f^+\Omega_f^-} \in K_f$. By Lemma 6.6,

$$< \widetilde{I}_f^+, \widetilde{I}_f^- >= c_k < f, f >_\Gamma .$$

And $(2\pi i) * c_k \in K_f$ (almost, I am off by i, need to check my work back in Prop [?]). $\square$

Example 6.7. For weight 2 forms, these periods are not so mysterious at least from a computational point of view. We say a divisor $D = \sum a_n r \to s$ is in the $+$-part if $\iota(D) := \sum a_n -r \to -s$ is $\Gamma_0(N)$ to D. For example, $\{0 \to \infty\}$ is clearly in the $+$-part. In the weight 2 case, we have

$$\widetilde{I}_f(D) = \widetilde{I}_f^+(D) + \widetilde{I}_f^-(D) = \widetilde{I}_f^+(D)$$

if D is in the $+$-part. That means that $\widetilde{I}_f(D)$ will always be a K_f-multiple of Ω_f^+. In particular, $L(f,1)$ will always be a K_f-multiple of Ω_f^+. We can similarly define a $-$-part to compute a K_f-multiple of Ω_f^-.

7. L-FUNCTIONS

As we saw in our example, special values of L-functions can be written in terms of periods, and thus are encoded in the modular symbol. In this section, we give a formula for certain L-values in terms of modular symbols. We prove a well-known formula of Birch and Manin for the "algebraic" part of the L-value.

Let $f \in S_k(\Gamma_0(N), \mathbb{C})$ and let $\chi : (\mathbb{Z}/m\mathbb{Z})^* \to \mathbb{C}^*$ be a primitive Dirichlet character modulo m. (Note that we set $\chi(n) = 0$ if $(n, m) \neq 1$.) Define a function in the complex variable s by

$$L(f, \chi, s) := \sum_{n \geq 0} a_n \chi(n) n^{-s}.$$

where the a_n are the Fourier coefficients of f. This series converges for $\mathrm{Re}(s)$ sufficiently large and admits a meromorphic continuation to the entire complex plane. The classical proof of analytic continuation involves the study of the convergance of the Mellin transform of f.

Define

$$f_\chi(z) = \sum a_n \chi(n) q^n$$

where $q = e^{2\pi i z}$. The function f_χ has similar convergence properties to f, since the a_n satisfy same growth conditions. In fact, f_χ will be the Fourier expansion of some possible higher level cusp form so we can think of f_χ as a holomorphic on the $\mathcal{H}$ with well-defined periods. We will use the following classical formulation of the Mellin transform without proof:

$$L(f, \chi, s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty f_\chi(iy) y^{s-1} dy.$$

Definition 7.1. Let $\Psi \in MS_{\Gamma_0(N)}(V_k(F))$. For $a, m \in \mathbb{Z}, m \neq 0$, and $1 \leq j < k$,

$$\Psi[j, a, m] := \Psi\{\infty \to \frac{a}{m}\}(P_j|_{\left[\begin{smallmatrix} 1 & -\frac{a}{m} \\ 0 & 1 \end{smallmatrix}\right]}),$$

where $P_j(X, Y) = X^{j-1} Y^{k-j-1}$.

I know it is not so good that we have to keep introducing more and more notation. One can think of $\Psi[j, a, m]$ as being the "special value"' of the modular symbol at j and twisted by $\frac{a}{m}$, if that helps.

Proposition 7.2. *The value of $\Psi[j, a, m]$ only depends on the class of $a \in \mathbb{Z}/m\mathbb{Z}$. That is, For any $n \in \mathbb{Z}$,*

$$\Psi[j, a, m] = \Psi[j, a + nm, m].$$

Proof. We know that modular symbol Ψ is invariant under action of $\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]$. From this, we are able to deduce that

$$I_f[j,a,m] = I_f\{\infty \to \frac{a+m}{m}\}\left(P_j|_{\left[\begin{smallmatrix}1&-\frac{a+m}{m}\\0&1\end{smallmatrix}\right]}\right) = I_f[j,a+m,m].$$

□

Define the Gauss sum

$$\tau(\chi,n) := \sum_{a \bmod m} \chi(a)e^{\frac{2\pi ina}{m}}.$$

Lemma 7.3. *(pre-Birch Lemma) Let χ be a primitive Dirichlet character modulo m, then*

$$\tau(\chi,n) = \overline{\chi(n)}\cdot\tau(\chi,1).$$

Proof. For $(n,m)=1$, we have

$$\begin{aligned}\sum_{a \bmod m} \chi(a)e^{\frac{2\pi ina}{m}} &= \sum_{a \bmod m} \chi(n^{-1}a)e^{\frac{2\pi ia}{m}}\\ &= \chi(-n)\sum_{a \bmod m} \chi(a)e^{\frac{2\pi ia}{m}} = \overline{\chi(n)}\cdot\tau(\chi,1)\end{aligned}$$

where n^{-1} is taken in $(\mathbb{Z}/m\mathbb{Z})^*$. Assume $(n,m)=d\neq 1$ and write $n=dj$. Then,

$$\sum_{a \bmod m} \chi(a)e^{\frac{2\pi ina}{m}} =$$

□

Lemma 7.4 (Birch Lemma)**.** *With conditions as above,*

$$f_{\overline{\chi}}(z) = \frac{1}{\tau(\chi,1)}\sum_{a \bmod m} \chi(a)f(z+\frac{a}{m}).$$

Proof. We work with the Fourier series.

$$\begin{aligned}\sum_n \overline{\chi}(n)a_nq^n &= \frac{1}{\tau(\chi,1)}\sum_n \tau(\chi,n)a_nq^n\\ &= \frac{1}{\tau(\chi,1)}\sum_n\sum_{a \bmod m} \chi(a)e^{\frac{2\pi ina}{m}}a_nq^n\\ &= \frac{1}{\tau(\chi,1)}\sum_{a \bmod m} \chi(a)\sum_n a_ne^{2\pi in(z+\frac{a}{m})}\\ &= \frac{1}{\tau(\chi,1)}\sum_{a \bmod m} \chi(a)f(z+\frac{a}{m}).\end{aligned}$$

□

Corollary 7.5. *Assume now that χ is a quadratic character so $\chi = \overline{\chi}$. Then,*

$$\widetilde{I}_{f_\chi}[j,0,1] = \frac{1}{\tau(\chi,1)} \sum_{a \ mod \ m} \chi(a)\widetilde{I}_f[j,a,m].$$

Proposition 7.6 (Prop 1.3 in [BDInv])**.** *Let $1 \le j \le k-1$ be an integer and suppose that χ is a primitive quadratic Dirichlet character such that $\chi(-1) = (-1)^{j-1}w_\infty$, where $w_\infty \in 1,-1$ is the sign at infinity. Then, the expression*

$$L^*(f,\chi,j) := \frac{(j-1)!\tau(\chi,1)}{(-2\pi i)^{j-1}\Omega_f^{w_\infty}} L(f,\chi,j)$$

belongs to K_f, and

$$L^*(f,\chi,j) = \sum_{a=1}^{m} \chi(a) I_f^{w_\infty}[j,a,m].$$

Proof. We first claim that

$$\widetilde{I}_{f_\chi}[j,0,1] := \widetilde{I}_{f_\chi}\{\infty \to 0\}(P_j) = \frac{(j-1)!}{(-2\pi i)^{j-1}} L(f,\chi,j).$$

This will follow directly from applying Mellin transform to f_χ. We compute

$$\begin{aligned}
\widetilde{I}_{f_\chi}\{\infty \to 0\}(P_j) &= -(2\pi i)\int_0^\infty f_\chi(z)z^{j-1}dz \\
&= -(2\pi i)\int_0^\infty f_\chi(iy)(iy)^{j-1}(dx+idy) \\
&= -(2\pi)(i^{j+1})\int_0^\infty f_\chi(iy)y^{j-1}dy \\
&= -(2\pi)(i^{j+1})\frac{\Gamma(j)}{(2\pi)^j} L(f,\chi,j) \\
&= \frac{(j-1)!}{(-2\pi i)^{j-1}} L(f,\chi,j)
\end{aligned}$$

By Corollary 7.5, $\widetilde{I}_{f_\chi}[j,0,1]$ can be written in terms of $\widetilde{I}_f$. Some elementary simplification shows that

$$\Omega_f^{w_\infty} L^*(f,\chi,s) = \sum_{a \text{ mod } m} \chi(a)\widetilde{I}_f[j,a,m]$$

All that remains then is to show that left-hand side is "in" the w_∞ part of the modular, in which case, dividing by Ω^{w_∞} yields something in K_f. One can work it all out from the following formula, whose proof I leave to the reader

$$\iota(\widetilde{I}_f)[j,a,m] = (-1)^{j-1}\widetilde{I}_f[j,-a,m].$$

From here, one deduces that the "action" of ι on $\sum_{a \text{ mod } m} \chi(a)\widetilde{I}_f[j,a,m]$ is given by multiplication by $\chi(-1) * (-1)^{j-1}$. If $\omega_\infty = 1$, it is in the plus space, otherwise, the minus space. ☐

8. Appendix: Local Systems and π_1-Modules

Let Z be a locally path-connected and locally simply connected topological space. Choose a basepoint $z_0 \in Z$.

Proposition 8.1. *There is an equivalence of categories between local systems on Z and $\pi_1(Z, z_0)$ representations. If M is a π_1-modules, we denote the corresponding sheaf by $\widetilde{M}$.*

Proposition 8.2. *Let Z be a nice topological space as above. Assume that the univeral cover $\widetilde{Z}$ of Z has vanishing sheaf cohomology for all constant sheaves (for example, if $\widetilde{Z}$ is contractible). Then, for any $\pi_1(Z, z_0)$-module M, the maps $H^i(\pi_1(Z, z_0), M) \to H^i(Z, \widetilde{M})$.*

Proof. This is in Appendix F of [Con]. □

We will apply the lemma in two situations, in both cases, $\widetilde{Z} = \mathcal{H}$ which is a contractible space.

Example 8.3 (Example/Proposition)**.** Let Δ^* be the punctured unit disc. Then, the universal cover is $\mathcal{H}$, where the map is given by $e^{2\pi i z}$. A local system on Δ^* is the same as an action of $\mathbb{Z}$ as a additive group which is determined by where the generator 1 goes. This is the monodromy operator. The cohomology with coefficient in $\widetilde{M}$ is isomorphic to the group cohomology $H^i(\mathbb{Z}, M)$.

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